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COMBINED OPERATIONS WITH AND WITHOUT AFTERBURNING FOR MINIMUM FUEL CONSUMPTION IN LEVEL FLIGHT

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SUMMARY

The present report contains a preliminary analysis of combined operations with and without afterburning with regard to maneuvers of minimum fuel consumption. The simple case of level flight is investigated and the optimizing condition determined. A general procedure is developed for computing the special Mach number at which the transition from nonafterburning operations to afterburning operations must occur. It is shown that the altitude-Mach number plane can be divided into a number of basic regions, in each of which a preferred mode of operation exists for the engine. Several numerical examples are included illustrating the general theory and supplying a tangible proof of the minimal character of the solution.

INTRODUCTION

An important characteristic of some modern types of turbojet engines is that they embody that device of thrust augmentation which is commonly known as afterburning. As a result of the low fuel-to-air ratios used in turbojet engines, the products of combustion leaving the turbine contain enough unburned air to support further combustion. By an appropriate injection of fuel into the tail pipe the thrust can be considerably increased at the expense, however, of a decrease in the overall efficiency of the engine. In a typical case the afterburner increases the thrust by as much as 40 percent at $M = 0$, 70 percent at $M = 1$ and 100 percent at $M = 2$. The rate of fuel consumed per unit time (product of thrust times specific fuel consumption), however, increases by approximately 160 percent at $M = 0$, $M = 1$, and $M = 2$.

Clearly, the afterburning device is of interest in all those cases where superperformances are in order. For instance, the maximum speed of an aircraft can be considerably improved over short periods of time by the use of methods of thrust augmentation; analogously, the time necessary for a typical fighter interceptor to accelerate and climb can be shortened to a substantial extent.

In the present report, problems of minimum fuel consumption are considered and the possibilities arising from combined operations with and without afterburning explored. The key idea is that, for an aircraft which must be transferred from one condition of flight to another, any increase in energy height $\left(h_e = h + \dfrac{V^2}{2g}\right)$ may become rather expensive in terms of fuel if a portion of the trajectory is to be flown without afterburning in the so-called critical region (fig. 1). The latter, within the context of the present report, is approximately defined as that region of the altitude-Mach number plane which is in immediate contact with the geometrical locus of the points where $T - D = 0$, T being the thrust without afterburning and D the drag. As is known the drag D splits into zero-lift drag and induced drag. The latter is computed by approximating the equation of motion on the normal to the flight path as $L - W = 0$.

Starting with a simple physical scheme, the problem of the acceleration in level flight from one velocity to another is investigated. This investigation was conducted at Purdue University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

a	speed of sound, ft sec^{-1}
c	specific fuel consumption, sec^{-1}
C_D	drag coefficient
C_L	lift coefficient
D	drag, lb
g	acceleration of gravity, ft sec^{-2}
h	altitude above sea level, ft
K	ratio of induced drag coefficient to square of lift coefficient
L	lift, lb
M	Mach number

p — atmospheric pressure, lb ft^{-2}

q — weight of fuel consumed per unit time, lb sec^{-1}

R — air constant, ft^2 sec^{-2} $^{\circ}$R^{-1}

S — reference surface, ft^2

t — time, sec

T — thrust, lb

V — flight velocity, ft sec^{-1}

W — weight of aircraft, lb

W_f — weight of fuel consumed, lb

α — derivative of air temperature with respect to altitude, $^{\circ}$R ft^{-1}

γ — ratio of specific heat at constant pressure to specific heat at constant volume

λ — parameter defined by equation (22)

π — ratio of pressure at altitude h to pressure at the tropopause h_*

ρ — absolute density of air, lb ft^{-4} sec^2

τ — absolute temperature of air, $^{\circ}$R

Superscript

$(\bar{\ })$ — flight condition with afterburner operating

Subscripts

i — initial point

f — final point

t — transition point from operation without afterburner to operation with afterburner and vice versa

*	condition at tropopause
o	zero-lift condition or sea-level condition

FUNDAMENTAL HYPOTHESES AND EQUATIONS OF MOTION

In the present report a level flight trajectory flown over a short period of time is analyzed. As a consequence, the weight W of the aircraft is regarded as a constant in the equations of motion. The turbojet-powered airplane is thought of as a particle. The small angle between thrust vector and velocity vector is neglected. The aerodynamic lag is disregarded, that is, lift and drag forces are calculated as in unaccelerated flight.

It is assumed that the turbojet engine may develop two levels of thrust, one without afterburning and another one with afterburning. For the operation without afterburning the thrust T, the specific fuel consumption c, and the rate of fuel flow q are assumed arbitrary but specified functions of the following type:

$$T = T(V,h) \tag{1}$$

$$c = c(V,h) \tag{2}$$

$$q = cT = q(V,h) \tag{3}$$

where h is the flight altitude. Analogous hypotheses are accepted for the operation with afterburning (in general T and $\overline{T}$ are analytically different functions of velocity and altitude; the same remark applies to c and $\bar{c}$):

$$\overline{T} = \overline{T}(V,h) \tag{4}$$

$$\bar{c} = \bar{c}(V,h) \tag{5}$$

$$\bar{q} = \bar{c}\overline{T} = \bar{q}(V,h) \tag{6}$$

The equation of motion on the tangent to the flight path is written as:

$$T - D - \frac{W}{g}\frac{dV}{dt} = 0 \tag{7}$$

for the operation without afterburning and as

$$\overline{T} - D - \frac{W}{g}\frac{dV}{dt} = 0 \tag{8}$$

for the operation with afterburning. The attendant equation on the normal to the flight path is

$$L - W = 0 \tag{9}$$

In addition, the relationship between drag and lift is assumed to have the form

$$D = D(V,h,L) \tag{10}$$

OPTIMIZING CONDITION

The following functions are now defined:

$$\Phi = \frac{qW}{g(T - D)} \tag{11}$$

$$\bar{\Phi} = \frac{\bar{q}W}{g(\overline{T} - D)} \tag{12}$$

where Φ refers to engine operating without afterburning and $\bar{\Phi}$ to engine operating with afterburning. After accounting for equations (1), (3), (4), (6), (9), and (10), and eliminating the lift, one concludes that for a given weight the two functions Φ and $\bar{\Phi}$ have the form:

$$\Phi = \Phi(V,h) \tag{13}$$

$$\bar{\Phi} = \bar{\Phi}(V,h) \tag{14}$$

Assume now that the aircraft is to be transferred from an initial condition of flight (h,V_i) to a final condition of flight (h,V_f) at constant altitude. Assume also, for the sake of discussion, that the velocity interval (V_i,V_t) is flown without afterburning, while the velocity interval (V_t,V_f) is flown with afterburning[1]. The total weight of fuel consumed is given by:

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$$W_f + \bar{W}_f = \int_{t_i}^{t_t} q\,dt + \int_{t_t}^{t_f} \bar{q}\,dt = \int_{V_i}^{V_t} \Phi(V,h)dV + \int_{V_t}^{V_f} \bar{\Phi}(V,h)dV \tag{15}$$

Clearly, for given values of V_i and V_f equation (15) takes the form:

$$W_f + \bar{W}_f = f(V_t,h) \tag{16}$$

As a consequence, a necessary condition to the existence of an extremum for the weight of fuel consumed is that

$$\frac{\partial\left(W_f + \bar{W}_f\right)}{\partial V_t} = 0 \tag{17}$$

where the partial derivative is to be computed at constant altitude. By applying the general theorem of derivation under an integral sign, one obtains:

$$(\Phi - \bar{\Phi})_t = 0 \tag{18}$$

that is,

$$\left(\frac{cT}{T - D} - \frac{\bar{c}\bar{T}}{\bar{T} - D}\right)_t = 0 \tag{19}$$

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[1]The velocity V_t is the transition speed from one regime of operation to another.

With reference to the altitude-velocity plane or to the altitude-Mach number plane equation (19) defines the geometrical locus of the points where the transition from operation without to operation with afterburning must occur, if accelerated maneuvers of minimum fuel consumption are desired.

SOLUTION OF EQUATION DEFINING TRANSITIONAL SPEED

For a given altitude h the transition speed V_t from operation without afterburning to operation with afterburning is defined by equation (19). The latter must be generally solved by approximate methods. Nevertheless, by making suitable hypotheses about drag, thrust, and specific fuel consumption, considerable strides can be taken toward analytical solutions.

Drag Function

A parabolic relationship is now assumed between lift coefficient C_L and drag coefficient C_D:

$$C_D = C_{Do}(M) + K(M)C_L^2 \tag{20}$$

Both C_{Do} and K are assumed to depend on the Mach number M only.

After accounting for equations (9), (10), and (20) the drag function is written as:

$$D = \left(C_{Do}M^2 \frac{\pi}{\lambda} + \frac{K}{M^2} \frac{\lambda}{\pi}\right)W \tag{21}$$

where

$$\lambda = \frac{2W}{\gamma p_* S} \tag{22}$$

$$\pi = \frac{p}{p_*} \tag{23}$$

In the above equations p is the static pressure at altitude h and p_* the static pressure at the tropopause h_*.

Thrust Function

With the object of deriving simple solutions, the thrust is assumed to be the product of a function of the Mach number only times a function of the density ratio only. The first function is regarded as identical with the thrust at the tropopause. The second function is considered a power of the density ratio only:

$$T = T_*(M)\left(\frac{\rho}{\rho_*}\right)^x = T_*(M)\pi^{mx} \tag{24}$$

$$\overline{T} = \overline{T}_*(M)\left(\frac{\rho}{\rho_*}\right)^x = \overline{T}_*(M)\pi^{mx} \tag{25}$$

$$m = 1 + \frac{\alpha R}{g} \tag{26}$$

In the above equations $R = p/\rho\tau$ the air constant and α the derivative $d\tau/dh$ of the absolute temperature τ with respect to altitude h ($-0.003566\ {}^{\circ}R\ ft^{-1}$ for troposphere; $0\ {}^{\circ}R\ ft^{-1}$ for isothermal stratosphere). Typical values for the x exponent are: $x = 0.75$ for tropospheric flight and $x = 1$ for stratospheric flight. The m constant is $m = 0.8097$ for the troposphere and $m = 1$ for the isothermal region of the stratosphere.

Specific Fuel Consumption Function

With regard to the specific fuel consumption, the following function is assumed:

$$c = c_*(M)\left(\frac{\rho}{\rho_*}\right)^y = c_*(M)\pi^{my} \tag{27}$$

$$\bar{c} = \bar{c}_*(M)\left(\frac{\rho}{\rho_*}\right)^y = \bar{c}_*(M)\pi^{my} \tag{28}$$

where y is an appropriate constant (typical values: $y \doteq 0.15$ for tropospheric flight; $y = 0$ for stratospheric flight).

Generalized Solution

After accounting for equations (21) to (28), equation (19) can be rewritten[2] as

$$\pi^{1+mx} - A\pi^2 - B = 0 \tag{29}$$

where

$$A = \frac{C_{D_0} M^2}{\lambda \dfrac{T_*}{W}} \; \frac{\dfrac{\bar{C}_*}{C_*} - \dfrac{T_*}{\bar{\bar{T}}_*}}{\dfrac{\bar{\bar{C}}_*}{C_*} - 1} = A(M) \tag{30}$$

$$B = \frac{K\lambda}{M^2 \dfrac{T_*}{W}} \; \frac{\dfrac{\bar{C}_*}{C_*} - \dfrac{T_*}{\bar{\bar{T}}_*}}{\dfrac{\bar{\bar{C}}_*}{C_*} - 1} = B(M) \tag{31}$$

Stratospheric flight.- For flight in the isothermal region of the stratosphere $m = x = 1$ equation (29) can be solved in terms of the relative pressure:

$$\pi = \sqrt{\frac{B}{1 - A}} \tag{32}$$

As a consequence, the relationship between altitude and Mach number is written as:

$$h = h_* + \frac{R\tau}{2g} \log \frac{1 - A}{B} \tag{33}$$

Tropospheric flight.- For flight in the troposphere, the transitional speed equation must be solved by approximate procedures. Notice

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[2]The subscript t denoting transitional condition is now dropped, since there is no longer any possibility of ambiguity.

that, given the aircraft and the engine, equation (29) is represented by a family of straight lines in the AB plane, namely one straight line for each value of the pressure ratio π (fig. 2).

In the case where the transition speed V_t is to be calculated for several values of the altitude h, the following indirect procedure is appropriate:

(1) Select an arbitrary Mach number and compute the two functions $A(M)$ and $B(M)$;

(2) Enter into figure 2 and determine the pressure ratio corresponding to the established Mach number.

In view of the graphical operations involved in step (2), the above method yields an approximate value for the pressure ratio π_a. Nevertheless, the inherent error can be corrected by writing the exact solution of equation (29) in the form:

$$\pi = \pi_a(1 + \delta) \tag{34}$$

The correction term $\delta \ll 1$ can be computed by introducing equation (34) into equation (29) and linearizing the latter into:

$$\pi_a{}^{1+mx}\left[1 + \delta(1 + mx)\right] - A\pi_a{}^2(1 + 2\delta) - B \cong 0 \tag{35}$$

As a consequence, the correction δ is:

$$\delta \cong \frac{B + A\pi_a{}^2 - \pi_a{}^{1+mx}}{(1 + mx)\pi_a{}^{1+mx} - 2A\pi_a{}^2} \tag{36}$$

WEIGHT OF FUEL CONSUMED

Under the same hypotheses, as stated earlier, the below indicated expression can be derived for the weight of fuel consumed. Reference is made to the portion of the flight path flown without afterburning:

$$\frac{W_f}{W} = \frac{a_* C_{*_o}}{g} \pi^{my} + \frac{1-m}{2} \int_{M_i}^{M_t} \frac{\frac{C_*}{C_{*_o}} \frac{T_*}{T_{*_o}} dM}{\frac{T_*}{T_{*_o}} - \frac{W}{T_{*_o}} \left[\frac{C_{Do} M^2}{\lambda} \pi^{1-mx} + \frac{K\lambda}{M^2} \pi^{-(1+mx)} \right]} \tag{37}$$

In the above equation a_* denotes speed of sound calculated at $h = h_*$, c_{*_o} specific fuel consumption at $h = h_*$ and $M = 0$, and T_{*_o} thrust at $h = h_*$ and $M = 0$. An analogous expression can be derived for the portion of the flight path flown with afterburning.

UNACCELERATED LEVEL FLIGHT EQUATION

It is of interest to note that the general procedure developed earlier for solving the transitional speed equation can also be used for solving the equation of the unaccelerated level flight for both the nonafterburning case and the afterburning case.

With reference to the nonafterburning case, the steady flight at constant altitude is defined by equations (9), (10), and by the following additional expression

$$T - D = 0 \tag{38}$$

Under the stated hypotheses for the drag and thrust functions, equation (38) can be reduced to the form represented by equation (29). The two functions A and B, however, modify into:

$$A = \frac{C_{Do} M^2}{\lambda \frac{T_*}{W}} \tag{39}$$

$$B = \frac{K}{M^2} \cdot \frac{\lambda}{\frac{T_*}{W}} \tag{40}$$

Concerning the stratospheric case, equations (32) and (33) are still valid, with A and B defined by equations (39) and (40). For the tropospheric case the general procedure of the section entitled

"Tropospheric flight" is also valid with particular reference to the use of figure 2 and equations (34) to (36).

Analogous remarks refer to the solution of the unaccelerated level flight equation with thrust augmentation, the only difference being that T and T_* must be replaced with $\overline{T}$ and $\overline{T}_*$.

DISCUSSION OF RESULTS

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With the object of illustrating the previous theory, a numerical example is carried out for the below indicated set of conditions.

The aircraft is represented by

$$\lambda = 0.233 \qquad C_{Doo} = 0.02 \qquad K_o = 0.212 \tag{41}$$

the assigned value for λ corresponding to a wing loading $W/S = 80\ \mathrm{lb\ ft^{-2}}$. The symbols C_{Doo} and K_o denote values of C_{Do} and K calculated at $M = 0$. The two ratios C_{Do}/C_{Doo} and K/K_o are represented in figure 3 as functions of the Mach number.

The engine is represented by

$$\frac{T_{*o}}{W} = 0.235 \qquad \frac{c_{*o}a_*}{g} = 0.0069 \tag{42}$$

$$\frac{\overline{T}_{*o}}{T_{*o}} = 1.4 \qquad \frac{\overline{C}_{*o}}{C_{*o}} = 1.9 \tag{43}$$

The two ratios T_*/T_{*o} and $\overline{T}_*/\overline{T}_{*o}$ are plotted in figure 4 as functions of the Mach number M; the other two ratios C_*/C_{*o} and $\overline{C}_*/\overline{C}_{*o}$ are indicated in figure 5.

Starting from the above data equation (29) is solved at various altitudes for three specific cases: best transitional speed, unaccelerated level flight without afterburning, and unaccelerated level flight with afterburning. The results of numerical computations are indicated in figure 6 in the altitude-Mach number plane.

Notice that the three curves $T - D = 0$, $\overline{T} - D = 0$ and $\Phi - \overline{\Phi} = 0$ split the hM plane into the four regions A, B, C, and D endowed with the following properties:

(1) In the regions A and B accelerated flight occurs both with afterburning and without afterburning.

(2) In region C accelerated flight is possible only with afterburning.

(3) In region D only decelerated flight occurs.

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If accelerated maneuvers of minimum fuel consumption are desired, the following rules must be kept in mind:

(1) All portions of the flight path which belong to the region A must be flown without afterburning.

(2) All portions of the flight path which belong to regions B and C must be flown with afterburning.

(3) The switching from one regime of operation to another must occur along the line $\Phi - \overline{\Phi} = 0$.

Specific Examples for Prescribed Set of End Conditions

Some specific numerical examples are now presented (fig. 6).

(1) Assume that the flight altitude is $h = 40{,}000$ feet, that the initial Mach number is $M_i = 0.8$ and that the final Mach number is $M_f = 2$. The initial point I belongs to region B and the final point F to region C. As a consequence, the optimum path IF is to be entirely flown with afterburning (solid line of fig. 6).

(2) Consider the case where the flight altitude is $h = 30{,}000$ feet. Assume $M_i = 0.6$ and $M_f = 2$. The initial point I belongs to region A and the final point F to region C. The optimum path ITF includes a portion IT flown without afterburning (dotted line) and a portion TF flown with afterburning (solid line), the transition occurring at Mach number $M_t = 0.95$. The associated distribution of thrust is indicated in figure 7.

(3) Consider the case where the flight altitude is $h = 10{,}000$ feet, $M_i = 0.4$, and $M_f = 0.8$. Both the initial point I and the final point F belong to the region A. As a consequence, the optimum path IF is to be entirely flown without afterburning (dotted line of fig. 6).

Numerical Check of Minimal Character of Solution

In order to verify the minimal character of the solution, the second example of the preceding section is considered $h = 30{,}000$ feet, $M_i = 0.8$, and $M_f = 2$. Transition Mach numbers other than the optimum $M_t = 0.95$ are assumed and the function:

$$\frac{W_f + \overline{W}_f}{W} = f(M_t) \tag{44}$$

is computed and plotted in figure 8. The latter points out the following concepts:

(1) The transitional solution $\Phi - \bar{\Phi} = 0$ minimizes the fuel consumed.

(2) A shifting in the transition Mach number with respect to the optimum one may cause a considerable increase in fuel consumed if the transition point penetrates into the so-called critical region, as defined in figure 1.

CONCLUSIONS

Combined operations with and without afterburning are analyzed in connection with the problem of accelerating an aircraft in level flight from one velocity to another.

A general procedure is developed for computing the transitional speed; that is, the velocity at which the switching from nonafterburning operations to afterburning operations must occur for minimum fuel consumption.

Several regions are detected in the altitude-Mach number domain, in each of which a preferred mode of operation exists for the engine. Numerical examples are included, illustrating the general theory and confirming the minimal character of the solution.

Purdue University,
Lafayette, Ind., April 22, 1958.

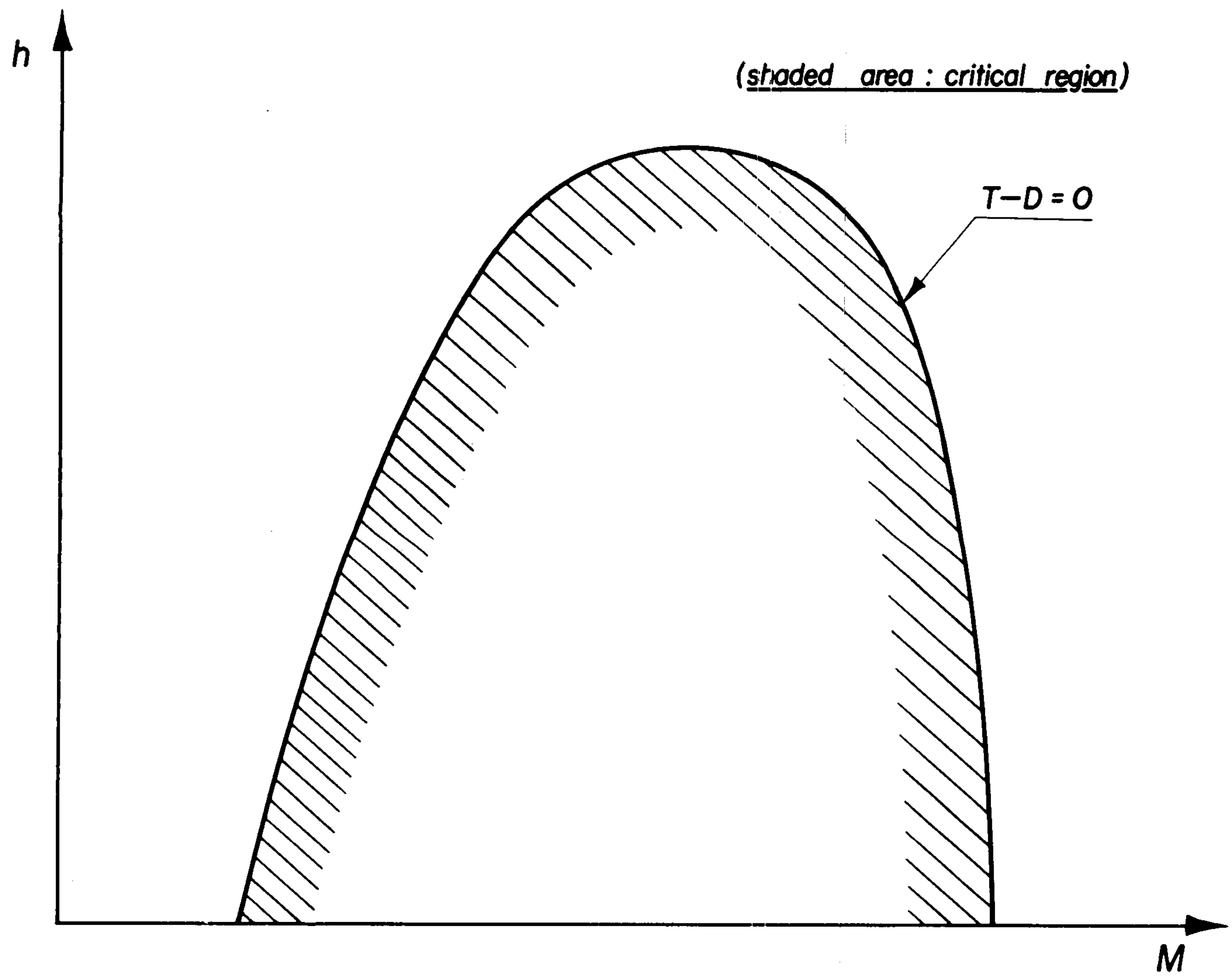

Figure 1.- Approximate definition of critical region for operations without afterburning.

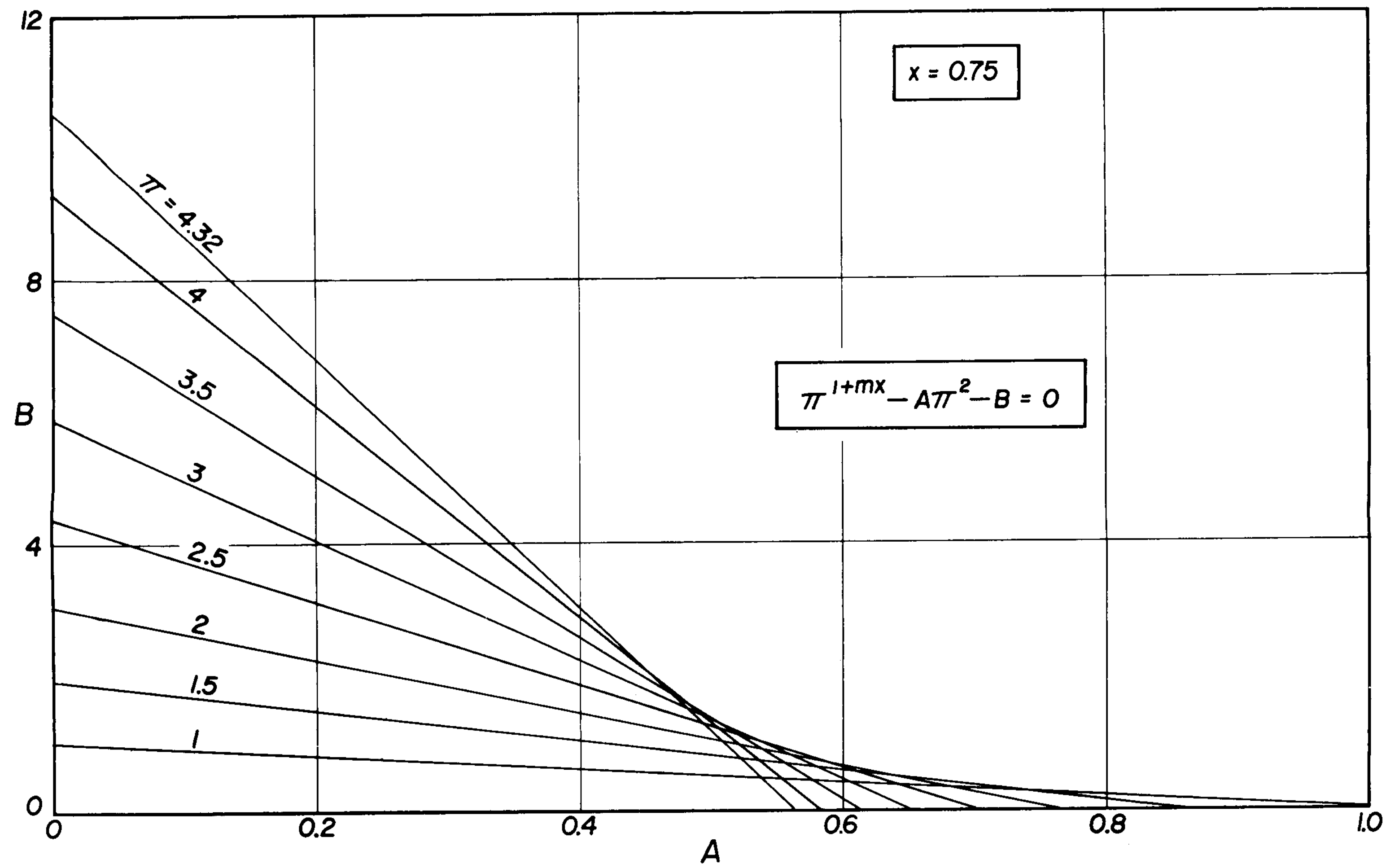

Figure 2.- Generalized graph for solution of transitional speed equation and of unaccelerated level flight equation.

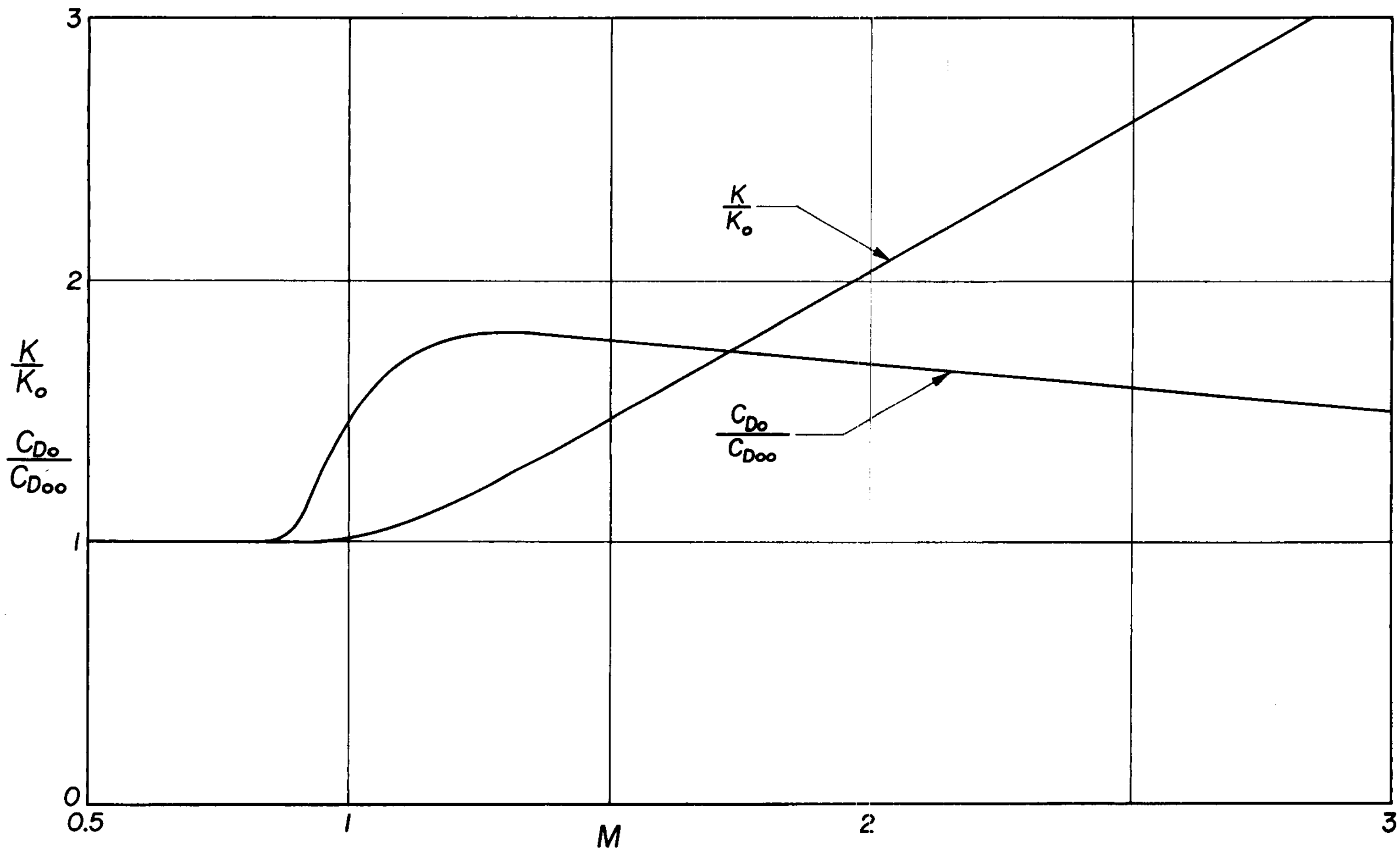

Figure 3.- Variation of zero-lift drag coefficient and of induced drag factor with Mach number (hypothetical aircraft).

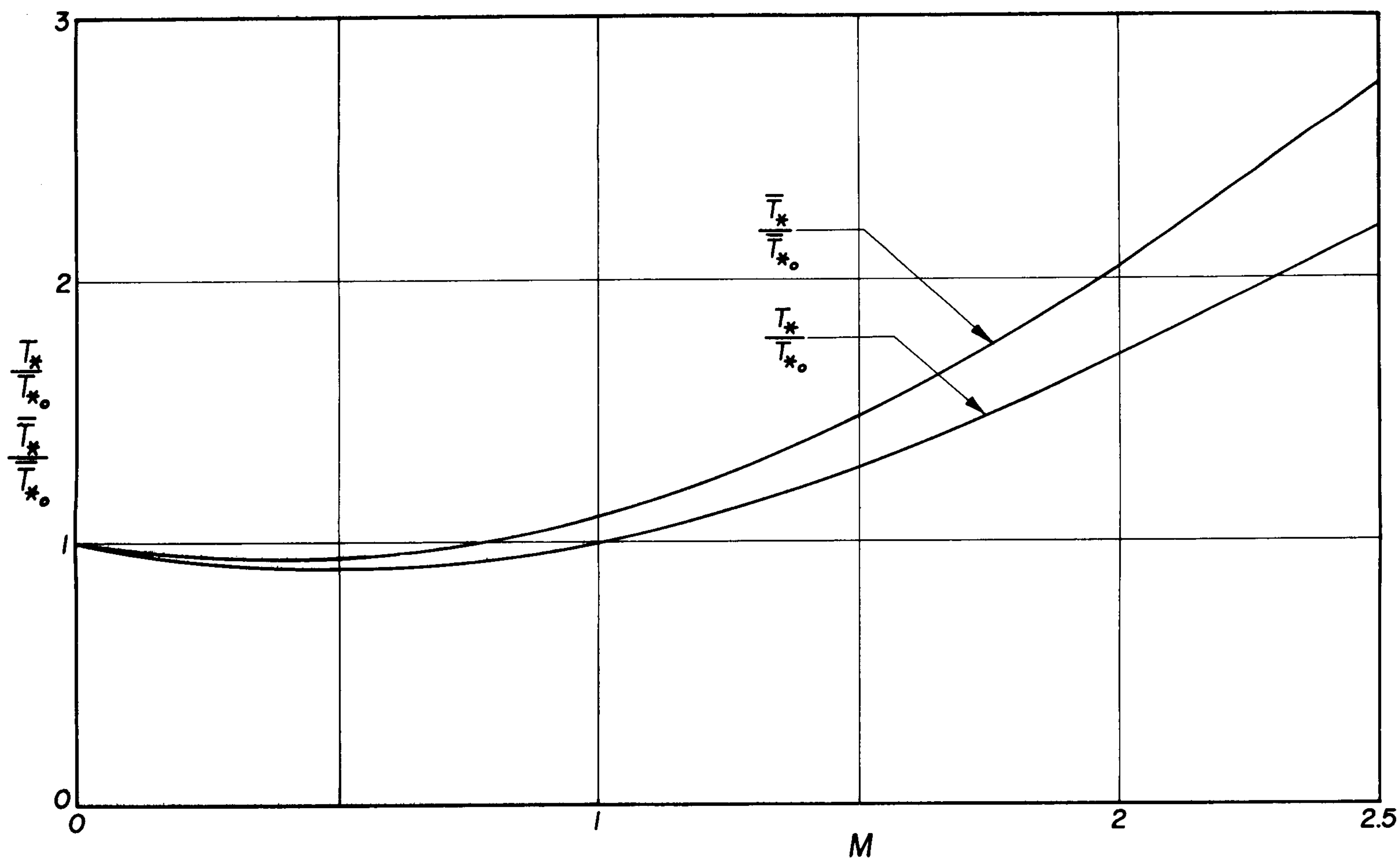

Figure 4.- Thrust at tropopause as a function of Mach number for engine operating with and without afterburner (hypothetical turbojet).

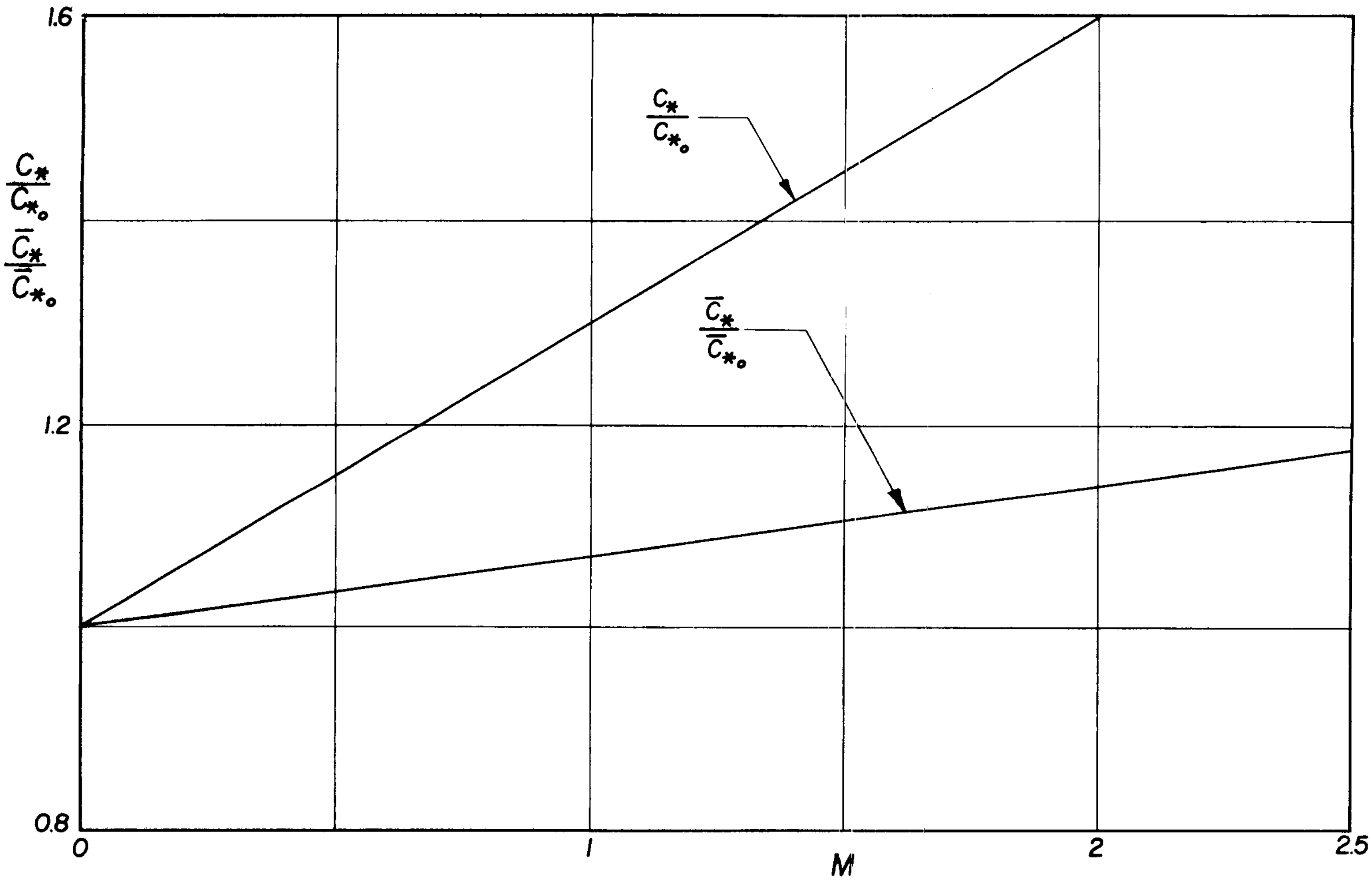

Figure 5.- Specific fuel consumption at tropopause as a function of Mach number for engine operating with and without afterburner (hypothetical turbojet).

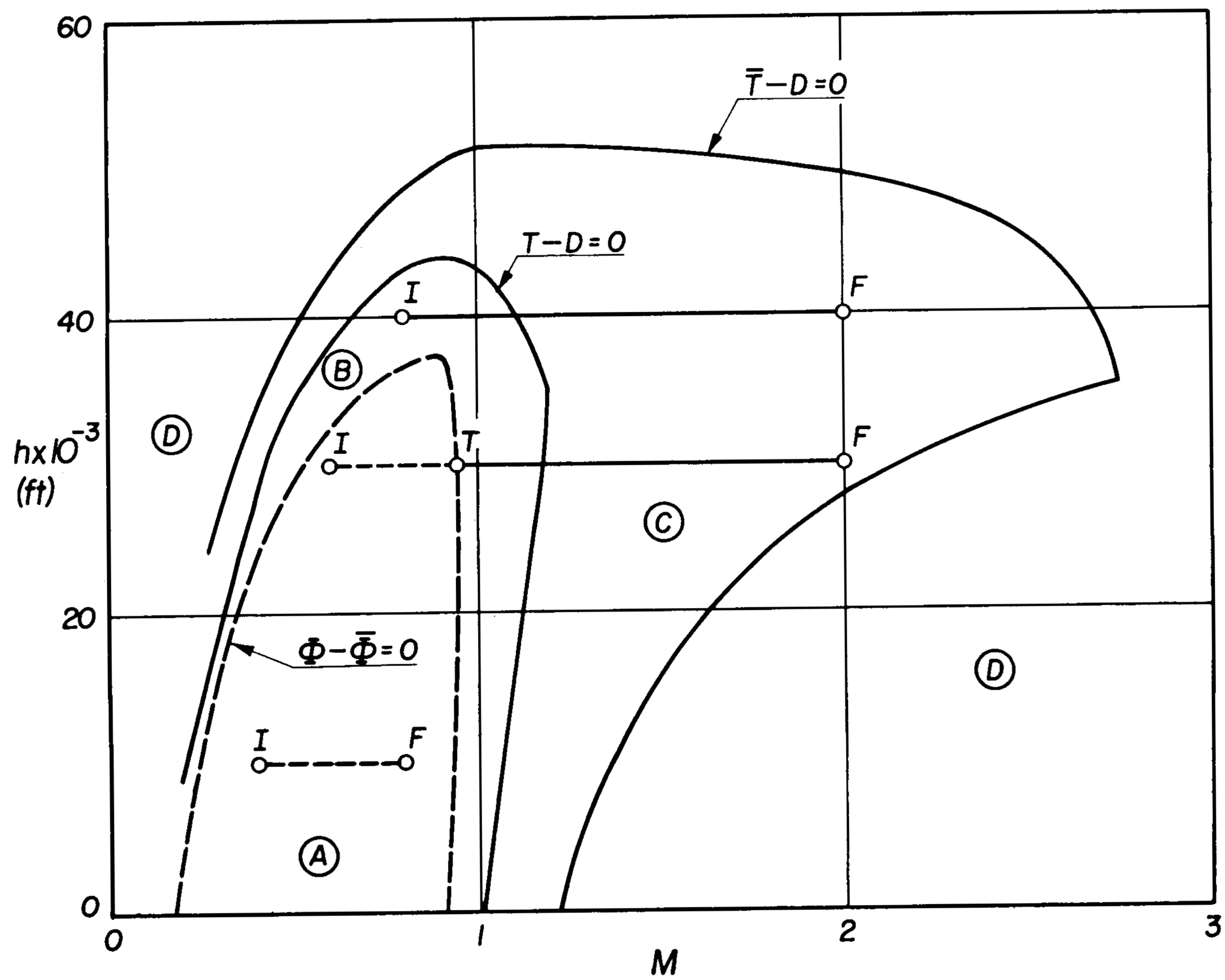

Figure 6.- Basic regions of altitude-Mach number plane.

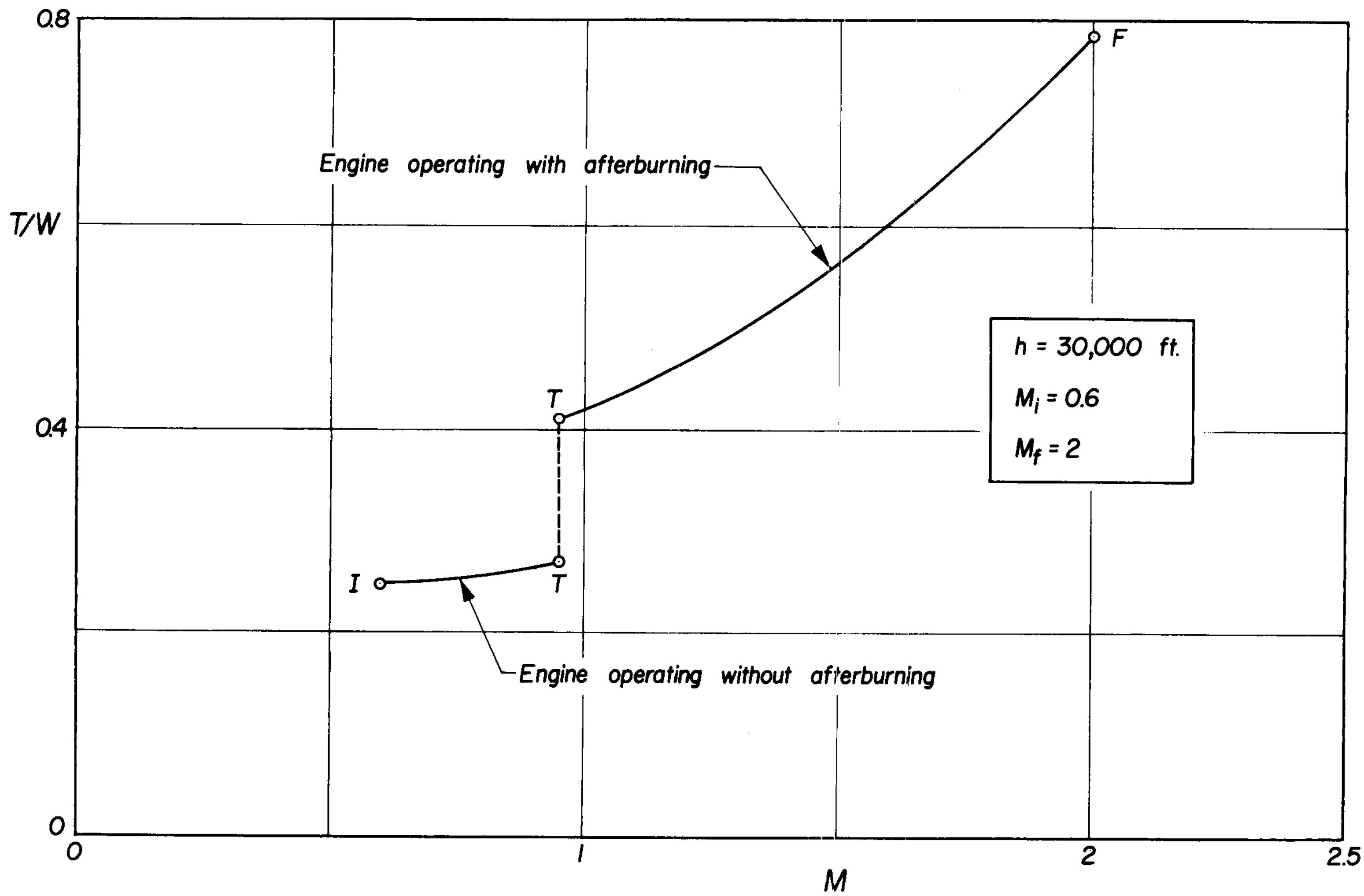

Figure 7.- Optimum distribution of thrust in a typical example.

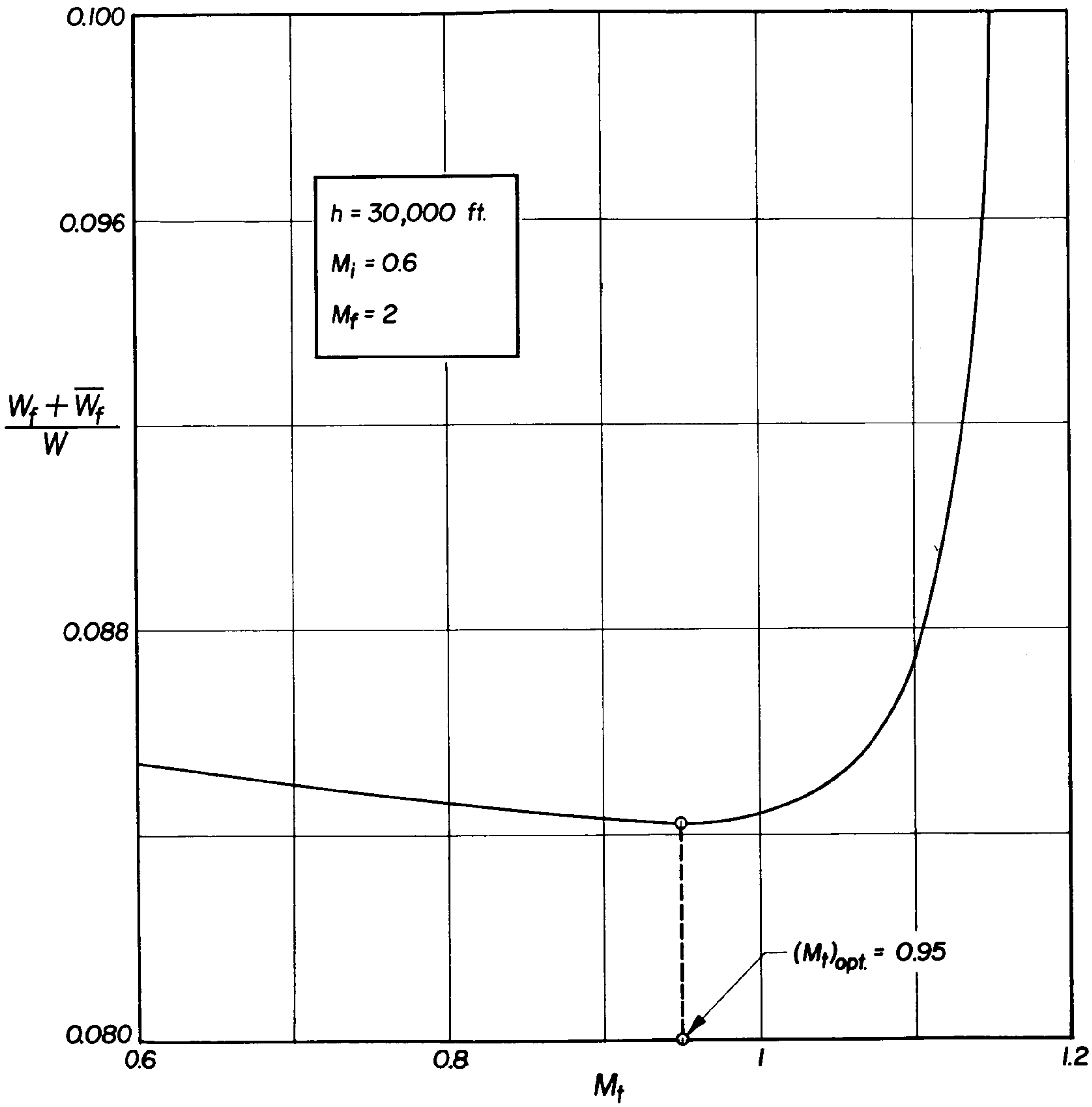

Figure 8.- Influence of transition Mach number on fuel consumed while accelerating.